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2009 J. Phys.: Condens. Matter 21 325702

(http://iopscience.iop.org/0953-8984/21/32/325702)

IOP PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **21** (2009) 325702 (6pp) doi:10.1088/0953-8984/21/32/325702

The rate of thermal expansion of a thin metallic slab of arbitrary shape

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Received 19 April 2009, in final form 24 June 2009
Published 20 July 2009
Online at stacks.iop.org/JPhysCM/21/325702

Abstract

In a previous paper the rate of thermal expansion of a long, slender insulating bar has been worked out. Our present aim is to extend that work to the thermal expansion rate of not only a long *metallic* bar, but to further generalize it to a thin metallic *slab of arbitrary shape*. Assuming that the thickness of the slab is small compared to the linear dimension of its area we again take advantage of the two distinct, disparate timescales to turn the familiar problem of thermal expansion into a time-dependent problem of the *rate* of the expansion. Based on the previously established finite physical momentum of an acoustic phonon when translational invariance is broken, we show that the combined pressure of the phonons and the free electrons due to their outward momenta would suffer a Doppler reduction as the specimen expands upon heating. This Doppler reduction gives rise to damping of the expanding motion, thus yielding as a first result the time of thermal expansion of a long slender metal bar. The generalization to the important case of a thin metallic slab of *any* shape is then worked out in detail before a concluding section containing a long physical discussion and summary.

1. Introduction

The rate of thermal expansion of a long, thin, insulating bar has been derived in a recent publication [1] to which the present paper should be considered as a sequel.

How fast a long bar or a thin disk expands when heated is not only of scientific interest but also of practical significance. As a concrete example, take the recent proposal for an x-ray free electron laser oscillator with an energy-recovery linac [2, 3]. One of the main components is an x-ray optical cavity with *thin crystal plates* (0.05–0.1 mm) of diamond or sapphire used as Bragg diffracting mirrors for x-rays. The incident x-ray pulse about 1 ps long will heat the crystal due to photoabsorption of about 2% of the incident pulse, with an absorption time $\sim$1 ps. *How fast* the temperature rise ΔT leads to the *thermal expansion of the crystal* is of great importance, for this expansion is related to the change of the lattice parameter, which is of crucial importance for the Bragg scattering from the crystal. Other relevant interesting work such as that on an infrared radiation pyrometer for transient temperature measurement and a photoelectric installation for micrometric length-change measurement can be found [4].

Generally, in this age of nanophysics, thermodynamic objects and the means to treat them must be adapted to very small sizes and very high speeds of temperature modulation for which the time of thermal expansion is intimately involved. Although thermal expansion is a familiar topic in standard solid-state texts [5] there seems to be very little work on how fast the expansion proceeds. This is probably because it belongs to the more complex realm of time-dependent statistical *dynamics* of thermal non-equilibrium, involving various physical aspects of vastly different length and time scales as discussed in detail previously [1]. An approach that can satisfactorily encompass all these physical features, without the encumbrance of impossibly complicated mathematical and/or numerical tools, had to be found. This is what was accomplished for the problem of the thermal expansion rate of a long, thin insulating bar. Our present goal is to extend that study to the problem pertaining to a *thin metallic or insulating slab of any shape*. For this purpose we shall follow the pattern of our previous work, but adapt it to include the combined influence of both the phonons and the free electrons in a metal. For mathematical reasons this is done in two steps. The first step is the calculation for a slender metallic bar, followed in the second step by the generalization to a thin metallic slab of arbitrary shape, culminating in an explicit expression for the time and speed of the thermal expansion. It might be counterintuitive that, as it turns out, the added influence of the free electrons in metals turns out to make the damping to the expanding motion a little greater and hence the expansion time a little shorter than for insulators. The physical reason will be explained.

In section 2, based on the concept of the two distinct timescales—one for thermal-kinetic equilibrium, the other for thermal-mechanical equilibrium—in the context of the speed of thermal expansion as in [1], we turn a problem of time-independent thermal equilibrium of thermal expansion into a dynamic, time-dependent problem of the *rate* of thermal expansion. Section 3 recalls how a failed first attempt at finding the expansion rate was corrected by incorporating the concept of the physical momentum of an acoustic phonon (when translational invariance is broken) into the picture [6] and how its reflection from the *expanding* free boundary of the specimen leads to a Doppler-induced reduction of the phonon momentum and hence to the dissipation of energy. We also include, when compared to our previous work [1], the influence of the free electrons in a *metal* which is now combined with that of the phonons, culminating in an expression for the damping coefficient for metals $\eta_{\rm metal}$ in section 4.

In most practical applications it is the expansion time of a thin disc, rather than a long bar, that is of interest [2–4]. Therefore, in section 5 the time of thermal expansion of a slender long bar is generalized to that of a thin insulating or metallic slab of any shape. Indeed, our theoretical result is found to agree qualitatively and semiquantitatively with the experimental result of Tang *et al* [4]. Such results should be of practical importance in experimental applications that involve fast temperature modulations. The final section 6 ends with a physical discussion and summary.

2. Thermal equilibrium turned into thermal expansion

In a recent work [1] the following has been shown. (1) Counter to intuitive expectations the process of dynamical thermal expansion of a long, *thin* bar (establishing thermal-mechanical equilibrium) is generally much slower than the process of merely establishing temperature equilibration (thermal-kinetic equilibrium) upon heating because two vastly disparate timescales are involved. (2) Equally important is the recognition of the fact that the finite speed of thermal expansion hinges upon an earlier-derived result [6] that an acoustic phonon of wavevector $\vec{q} \neq 0$ carries a finite physical momentum that arises from anharmonicity, provided translational symmetry is broken. (1) can be easily established for a long, thin bar of thickness b and length L as long as $L \gg b$. For details on the physical reasoning or the mathematical steps references [1, 6] should be consulted.

Following the formulation of [1] we now start from the expression for the ambient pressure P of a metallic system at temperature T [5]:

$$P = -\left[\frac{\partial}{\partial V}\left(U_{\rm stat.latt} + \frac{1}{2}\sum_{\vec{q}}\hbar\omega_{\vec{q}}\right)\right]_T + P_{\rm ph} + P_{\rm el} \qquad (1)$$

where

$$P_{\rm ph} = -\left(\frac{\partial U_{\rm phonon}}{\partial V}\right)_S = -\sum_{\vec{q}}\left(\frac{\partial \hbar\omega_{\vec{q}}}{\partial V}\right)_S \langle n_{\vec{q}}(T)\rangle \qquad (2)$$

is the contribution to the pressure by the phonons, $\langle n_{\vec{q}}(T)\rangle$ being the thermal averaged number of $\vec{q}$-phonons. In the Grüneisen approximation, $(\partial\hbar\omega_{\vec{q}}/\partial V)_S = -\gamma_{\rm ion}\hbar\omega_{\vec{q}}/V$ so that

$$P_{\rm ph} = \frac{\gamma_{\rm ion}\sum_{\vec{q}}\hbar\omega_{\vec{q}}\langle n_{\vec{q}}(T)\rangle}{V} = \frac{\gamma_{\rm ion}U_{\rm ph}(T)}{V}. \qquad (3)$$

Similarly,

$$P_{\rm el} = -\left(\frac{\partial U_{\rm el}}{\partial V}\right)_S = -\sum_{\vec{k}}\langle n_{\vec{k}}(T)\rangle\left(\frac{\partial(\hbar\omega_{\vec{k}})}{\partial V}\right)_S \qquad (4)$$

$$= -\sum_{\vec{k}}\langle n_{\vec{k}}(T)\rangle(\hbar^2/2m)\left(\frac{\partial\left(\frac{n_x^2+n_y^2+n_z^2}{V^{2/3}}\right)}{\partial V}\right)_{n_x,n_y,n_z} \qquad (5)$$

$$\equiv -\sum_{\vec{k}}\langle n_{\vec{k}}(T)\rangle\left(-\gamma_{\rm el}\frac{\hbar\omega_{\vec{k}}}{V}\right) \qquad (6)$$

$$= -\sum_{\vec{k}}\langle n_{\vec{k}}(T)\rangle\left(-\frac{2}{3}\frac{\hbar\omega_{\vec{k}}}{V}\right) = \frac{2}{3}\frac{U_{\rm el}}{V} \qquad (7)$$

is the contribution to the pressure by the gas of free electrons in the metal. Here n_x, n_y, n_z are the quantum numbers specifying the discretized electron momenta which remain unchanged in any variations when the entropy S is kept constant. $U_{\rm el}$ is the total internal energy, with the equivalent electronic Grüneisen coefficient identified, here, as $\gamma_{\rm el} = 2/3$. Incidentally, we could similarly show that $\gamma_{\rm photon} = 1/3$ [4]. The first term in equation (1), on the other hand, is the contribution from the energy of the strained lattice in static equilibrium, including that of the zero-point fluctuations. Neglecting the small ambient (e.g., atmospheric) pressure P we see that equation (1) expresses the mechanical balance of the (positive) outward pressure of $(P_{\rm ph} + P_{\rm el})$ by the force of the strained atomic springs that tend to restore the atoms to their original configuration.

Although the above equation (1) is supposed to describe only the time-independent state of total thermal equilibrium at temperature T *after* the full thermal expansion has been attained, we may now take advantage [1] of (1) to mean that the temperature equilibration process or 'thermal-kinetic equilibrium' [7] has essentially been completed while the much slower thermal expansion process towards establishing 'thermal-mechanical equilibrium' is just starting, isothermally. Recalling that a temperature rise ΔT is accompanied simultaneously by an increased phonon pressure $\Delta P_{\rm ph}$ as well as an increased electron pressure $\Delta P_{\rm el}$, according to the kinetic definition of temperature [7] we may re-interpret this equation (1), originally for thermal equilibrium, as now describing a quiescent elastic lattice (a bar of cross-section A) being pulled subsequently outward by the increased pressures $(\Delta P_{\rm ph} + \Delta P_{\rm el})$ through numerous quasi-static stages of the sluggish dynamical expansion process. At each one of these stages the thermodynamic variables T, P and V can thus be assumed to have well-defined values. However, unlike the total thermal equilibrium, in which the two terms of equation (1) balance each other, the outward (phonon + electron) force of the second term would now be too large to be balanced by the restoring force of the strained atomic springs until the thermal expansion has finally completed. This is of course why the

system keeps expanding until thermal-mechanical equilibrium is reached. Furthermore, in the Grüneisen approximation it is well known that $(\partial\omega_{\vec{q}}/\partial V)_S = -\gamma_{\rm ion}\omega_{\vec{q}}/V$, $\gamma_{\rm ion}$ being the ionic Grüneisen coefficient, so that $\Delta P_{\rm ph} \propto \gamma_{\rm ion}\Delta T$, based on equation (2). Other anharmonic effects such as phonon–phonon coupling are neglected by treating the phonons as free bosons, rendering the harmonic lattice vibration modes still independent of each other [5]. Similarly $\Delta P_{\rm el} \propto \gamma_{\rm el}\Delta T$ based on equation (6).

3. Attempts at finding expansion time

For a long thin insulating bar of cross-section area A and length L with one end fixed (this could be the stationary center of mass), we assume that under the above mentioned pull at the free end by $(\Delta P_{\rm ph} + \Delta P_{\rm el})$ every spring between two neighboring atoms separated by lattice spacing a is stretched by the same amount $\langle x(t)\rangle$ at any time t. But what should be the equation that governs $\langle x(t)\rangle$? Now that the needed pieces of physics seem to be in place it is elementary to show that a simple work–energy consideration yields a Newton equation of motion for a *whole transverse layer* of atoms in the long bar:

$$\frac{{\rm d}^2\langle x(t)\rangle}{{\rm d}t^2} + \omega_{\rm e}^2\langle x(t)\rangle = a^2\frac{(\Delta P_{\rm ph} + \Delta P_{\rm el})}{m_{\rm eff}} \tag{8}$$

$$\text{where } \omega_{\rm e}^2 = \frac{k}{m_{\rm eff}} \equiv \frac{3k}{mN^2} \tag{9}$$

k being the atomic spring constant, m the atomic mass and N the number of transverse layers in the bar.

Unfortunately, the above equation leads only to an ever-oscillatory (at frequency $\omega_{\rm e}$) stretching denoted by $\langle x(t)\rangle$, rather than the anticipated final steady stretching as $t \longrightarrow \infty$. A crucial damping mechanism that would lead to a steady $\langle x(t)\rangle$ instead of an ever oscillating one is missing! This is now remedied by invoking (2) of section 2, namely, that an acoustic phonon of wavevector $\vec{q} \neq 0$ can couple via anharmonic interaction to another acoustic phonon of $\vec{q} = 0$ to form a composite phonon of the same $\vec{q}$, provided translational invariance is broken, as in the present case of a bar of finite length in contact with an external heat reservoir. It is via this mate of $\vec{q} = 0$ that it becomes possible for the composite itself to carry a physical momentum which has been shown to be

$$\vec{p}_{\vec{q}} = n\gamma_{\rm ion}\vec{q} \tag{10}$$

where n is the dimensionality. The presence of the Grüneisen coefficient $\gamma_{\rm ion}$ is the signature of anharmonicity.

We shall now make another attempt at the equation for $\langle x(t)\rangle$ by recalculating the pressure of the forward-rushing, bullet-like phonons and electrons impinging on the receding (with velocity $v_{\rm end}$) free end of an expanding bar, this time taking the forward and the reflected momenta of each phonon, as well as electron, into account.

For the sake of completeness we reproduce briefly the calculation for the phonons [1]. With the velocity of the forward phonon relative to the free end given by $c_{\rm s} - v_{\rm end}$, the acoustic phonon frequency $\omega_{\vec{q}}$ is Doppler shifted to $\omega'_{\vec{q}} = \omega_{\vec{q}}(c_{\rm s} - v_{\rm end})/c_{\rm s}$, where $c_{\rm s}$ is the sound velocity. Correspondingly the momentum of each such phonon is changed from $\vec{p}_{\vec{q}}$ of equation (10) to the relative momentum $\vec{p}'_q$ along the longitudinal direction:

$$\vec{p}'_q = \frac{1\hbar\gamma_{\rm ion}\omega'_{\vec{q}}}{c_{\rm s}} = \vec{p}_{\vec{q}}\left(1 - \frac{v_{\rm end}}{c_{\rm s}}\right). \tag{11}$$

Correspondingly the outward phonon pressure on the free end is easily shown to be Doppler shifted to

$$P'_{\rm ph}(T) = P_{\rm ph}(T) + \delta P_{\rm ph}(v_{\rm end}) \tag{12}$$

where $P_{\rm ph}(T)$ is given previously by equation (3) and

$$\delta P_{\rm ph}(v_{\rm end}) \approx -\frac{2v_{\rm end}}{c_{\rm s}}P_{\rm ph}(T) \tag{13}$$

is the Doppler-shift correction to the phonon pressure. Note that the latter is proportional to $v_{\rm end}$ with a negative sign. This provides part of the missing link needed for damping.

4. Influence of free electrons on expansion time

A similar calculation can be carried out for the electrons as follows. Implicitly we are again assuming that the electrons have first achieved temperature equilibration or thermal-kinetic equilibrium as in section 2. The free electrons cause their own outward pressure $P_{\rm elec}$ of equation (6) on the free end of the bar. As the quasi-stationary (compared to $v_{\rm F}$) lattice expands with velocity $v_{\rm end} > 0$ the end of the bar is receding from the impinging electrons that give rise to the electronic pressure. The net transfer of the forward momentum $p_x = mv_x > 0$ of the electrons to each unit area of the expanding end per unit time is

$$[(n_{\vec{p}}/2V)(v_x - v_{\rm end})\Delta t][2m(v_x - v_{\rm end})]\frac{1}{\Delta t} \tag{14}$$

$$= \frac{n_{\vec{p}}}{V}m(v_x - v_{\rm end})^2 \simeq \frac{n_{\vec{p}}}{V}mv_x^2\left(1 - \frac{2v_{\rm end}}{|v_x|}\right) \tag{15}$$

where the first square-bracketed factor represents the number of v_x-electrons impinging with a relative velocity $(v_x - v_{\rm end})$ on a unit area of the receding end of the bar in time Δt. The density of the $p_x > 0$ electrons is $n_{\vec{p}}/2V$. The second square-bracketed factor represents the difference between the effective incident momentum and the reflected momentum of each electron bouncing off the free end, i.e., the net momentum transfer from each such electron to that end. The Doppler-shifted electron pressure due to electrons of all possible $p_x = mv_x > 0$ is

$$P'_{\rm elec} = \frac{2}{V}\sum_{\vec{p}}\langle n_{\vec{p}}\rangle\frac{mv_x^2}{2}\left(1 - \frac{2v_{\rm end}}{|v_x|}\right). \tag{16}$$

When $v_{\rm end} = 0$ we obtain

$$P_{\rm elec} = \frac{2}{V}\sum_{\vec{p}}\langle n_{\vec{p}}\rangle\frac{mv_x^2}{2} = \frac{2}{3V}\sum_{\vec{p}}\langle n_{\vec{p}}\rangle\frac{mv^2}{2} = \frac{2U_{\rm elec}}{3V} \tag{17}$$

which agrees with equation (6). In the limit of $k_B T \ll \epsilon_F$ it is easy to show that $P_{elec} = (2/5) n_e \epsilon_F$, n_e being the electron density and ϵ_F the Fermi energy.

When $v_{end} > 0$ there is a correction $\delta P_{elec}(v_{end})$:

$$P'_{elec} = P_{elec} + \delta P_{elec}(v_{end}) \tag{18}$$

where

$$\delta P_{elec}(v_{end}) = -\frac{2}{V} \sum_{\vec{p}} \langle n_{\vec{p}} \rangle \frac{m v_x^2}{|v_x|} v_{end} \tag{19}$$

is the Doppler-induced correction to the electron pressure. For $k_B T \ll \epsilon_F$ we find

$$\delta P_{elec}(v_{end}) \simeq -\frac{1}{V} \sqrt{\frac{8m}{3}} \left\langle \sqrt{\epsilon_{\vec{p}}} \right\rangle v_{end} \tag{20}$$

$$= -\left(\sqrt{\frac{3m\epsilon_F}{2}} n_e \right) v_{end} = -\frac{5\sqrt{3}}{2} P_{elec} \left(\frac{v_{end}}{v_F} \right). \tag{21}$$

The total correction to the pressure due to the Doppler shift caused by $v_{end} > 0$ is obtained by combining equations (13) and (21):

$$\delta P(v_{end}) = \delta P_{ph}(v_{end}) + \delta P_{elec}(v_{end}) \tag{22}$$

$$= -\left[\frac{2}{c_s} P_{ph}(T) + \frac{5\sqrt{3}}{2 v_F} P_{elec}(T) \right] v_{end}. \tag{23}$$

As the temperature rises by ΔT the above Doppler-induced correction $\delta P(v_{end})$ would itself change to $\Delta[\delta P(v_{end})]$ due to the changes in the two partial pressures, $\Delta P_{ph}(T)$ and $\Delta P_{elec}(T)$. By using equations (3) and (6) we obtain

$$\Delta[\delta P(v_{end})] = -\left[\left(\frac{2}{c_s} \gamma_{ion} c_v^{ion} + \frac{5\sqrt{3}}{2 v_F} \gamma_{el} c_v^{el} \right) \Delta T \right] v_{end}. \tag{24}$$

Here c_v^{ion} is the specific heat of the ion lattice, and c_v^{el} is the specific heat of the electron gas in the metal. Addition of the above total $\Delta[\delta P(v_{end})]$ to $(\Delta P_{ph}(T) + \Delta P_{el}(T))$ on the right side of equation (8) leads to

$$\frac{d^2\langle x(t)\rangle}{dt^2} + \omega_e^2 \langle x(t)\rangle + \eta_{metal} \frac{d\langle x(t)\rangle}{dt} = a^2 \frac{(\Delta P_{ph} + \Delta P_{el})}{m_{eff}} \tag{25}$$

where the damping coefficient η_{metal} and hence the time for thermal expansion t_L^{metal} for a metal bar are identified as

$$\eta_{metal} = \frac{3}{L\rho} \left[\frac{2}{c_s} \gamma_{ion} c_v^{ion} + \frac{5\sqrt{3}}{2 v_F} \gamma_{el} c_v^{el} \right] \Delta T = \frac{2}{\Delta t_L^{metal}}. \tag{26}$$

The above equation constitutes the major result of the present investigation. The added influence of the free electrons on the expansion rate is clearly identified as coming from the second term inside the square bracket. It would make the damping a little greater and hence the expansion time a little shorter for metals than for insulators. As discussed in section 5, due mainly to the ratio of two speed factors $(c_s^{-1}/v_F^{-1}) \sim 10^2$ the influence of the electrons is much smaller than that of the phonons in η_{metal} at $T \lesssim 10$ K. The damping coefficient for an insulating bar can be obtained simply by setting γ_{el} to zero.

5. The speed of thermal expansion of a thin slab of any shape

So far we have confined our attention to the thermal expansion rate of a slender long bar. We shall now consider the process of thermal expansion of a thin slab of finite thickness b, but of arbitrary shape, as a result of a temperature rise ΔT in the surrounding heat reservoir. As in our previous work [1] or in section 1 we assume that b is small enough compared to the slab size that the time $\Delta t_T \approx b^2 c_V/(K\pi^2)$ for the slab to attain uniform temperature or 'thermal-kinetic equilibrium' is much shorter than that for the completion of thermal expansion for the slab in *all* directions parallel to its surface. For convenience we use the language appropriate for an insulating slab. The adaptation to a metallic slab by changing the language is immediate.

For mathematical convenience we first consider a square slab of side $2L = 2Na$ and thickness $b \ll L$. As a consequence, the thermal increase of the thickness in the direction normal to the slab is almost immediate and we may *then* focus on the expansion of the area of the slab. We further assume that the atoms are arranged in a square lattice with lattice spacing a on any layer parallel to the surface.

Suppose that, under the pulling force of the outward phonon pressure ΔP_{ph} that accompanies ΔT, every *spring* between two nearest neighboring atoms separated by a, either in the x-direction or the y-direction, is stretched by the same averaged amount $\langle \delta x(t) \rangle$ or $\langle \delta y(t) \rangle$. However, the atom at the center $(x = 0 = y)$ of the square is assumed fixed, that being the position of the center of mass. Indeed the $\langle \delta x(t) \rangle$-displacements of all the atoms on the y-axis are zero by reasons of symmetry. This is also true for the $\langle \delta y(t) \rangle$ displacements of all atoms on the x-axis.

Consider an x-line of spring-connected atoms all at the same height $y = n_y a$. Upon a temperature rise ΔT, every *atom* on this line will have one and the same transverse *displacement* $n_y \langle \delta y(t) \rangle$ while every *spring* on this line will have one and the same *stretching* $\langle \delta x(t) \rangle$. The latter means that the *atom* at $x = n_x a$ on the line will have a *displacement* $n_x \langle \delta x(t) \rangle$. Here n_x, n_y are integers counting from the origin, where $n_x = 0 = n_y$. Obviously similar statements can be made about a y-line of atoms.

We note that the longitudinal stretchings of all the springs on an x-line at a given height would not interfere with those of the springs on a neighboring x-line at a different height because the two lines remain parallel with (and at a uniform separation of $\langle \delta y(t) \rangle$ from) each other despite the thermal expansion. For the same reason the longitudinal stretchings of all the *springs* on a y-line would not interfere with those of another y-line. Furthermore, the *spring* stretchings of an x (y)-line would not interfere 'transversely' with the *spring* stretchings of a $y(x)$-line; all they could do is to change the separation between two parallel neighboring $y(x)$-lines. In other words, the square-lattice arrangement of the atoms in the slab can occupy an expanded area but the square shape is maintained. On the other hand, an arbitrary atom located at (x, y) may be displaced obliquely upon thermal expansion. For example, the x-component and the y-component of the displacement of

the atom at ($x = n_x a$, $y = n_y a$) are, respectively, $n_x\langle\delta x(t)\rangle$ and $n_y\langle\delta y(t)\rangle$ at time t. Since $\langle\delta x(t)\rangle$ is expected to be equal to $\langle\delta y(t)\rangle$ for the square lattice, upon thermal expansion the displacement of the atom at $(n_x a, n_y a)$ is directed radially outward at a slope of n_y/n_x from the center as physically expected.

It is clear from the above discussion that an x-line of atoms behaves just like the long slender bar treated in the main text, except that the cross-section area is now $A = 1a^2$. The coefficient of damping to the expanding motion for the slender bar is given by equation (26) with $\gamma_{\rm el}$ set to zero, namely,

$$\eta = 6\gamma c_v \Delta T/Lc_{\rm s}\rho \tag{27}$$

and the time for completion of the expansion is accordingly given by

$$\Delta t_L \equiv \frac{2}{\eta} = \frac{1}{9}\left(\frac{L}{c_{\rm s}}\right)\left(\frac{L}{\Delta L}\right) = \frac{1}{9}\left(\frac{L}{c_{\rm s}}\right)\frac{1}{\alpha\Delta T} \tag{28}$$

so that

$$v_{\rm exp} \equiv \frac{\Delta L}{\Delta t_L} = 9c_{\rm s}\left(\frac{\Delta L}{L}\right)^2 = 9c_{\rm s}(\alpha\Delta T)^2 \tag{29}$$

where $v_{\rm exp}$ is the speed of expansion, and γ is the Grüneisen coefficient, $c_{\rm s}$ the sound speed, c_v the heat capacity, ρ the mass density, α the thermal expansion coefficient, and $\Delta L = \alpha L\Delta T$ is the final value of the thermal expansion. Both results are seen to be independent of the cross-sectional area of the slender bar, since physically every line of atoms in the expanding bar meets the same fate. Hence these results are equally valid for the expansion of an x-line of atoms and, likewise, for a y-line of atoms. Each of these lines is of length L as reckoned from the fixed end located at the origin, the same as for the slender bar in the main text.

It is now easy to extend the above arguments to the times of expansion of a thin slab of any shape. For example, an arbitrary elliptic slab whose center of mass is at the origin may be marked off on a square slab also centered at the origin. The thermal displacement vector of the atom at ($x = n_x a$, $y = n_y a$) on the boundary of the marked ellipse, as on any other atoms on the square slab, has been shown to be along a radial direction from the origin, with Cartesian components $n_x\langle\delta x(t)\rangle$ and $n_y\langle\delta y(t)\rangle$ at time t. With the times for completing the stretchings $\langle\delta x\rangle$ and $\langle\delta y\rangle$ ($=\langle\delta x\rangle$) of the individual *springs* known, it follows that the time for completing the thermal displacement of any atom on the elliptic boundary is given by

$$\Delta t_L = \frac{1}{9}\left(\frac{L}{c_{\rm s}}\right)\frac{1}{\alpha\Delta T} \tag{30}$$

of equation (28), where the length L now stands for the radial distance $R = a\sqrt{n_x^2 + n_y^2}$ from the center of mass of the ellipse to that *atom* on the boundary. For example, the expansion time of a circular slab of radius R is simply given by the above formula with L replaced by R. A relevant physical discussion will be given after equation (32) in section 6.

The adaptation of the above formulae to the case of a metallic slab of any shape is straightforward. For example, our major result of $\eta_{\rm metal} = 2/\Delta t_L^{\rm metal}$ in equation (26) for a long metallic bar of length L can be taken to represent the time $\Delta t_L^{\rm metal}$ of thermal lengthening of the radial distance L as measured from the center of mass to any point on the boundary of a metallic slab of any shape in the expansion. A numerical estimate of $\Delta t_L^{\rm metal}$ can be immediately made for any metallic slab when the material parameters such as ρ, $\gamma_{\rm ion}$, $\gamma_{\rm el}$, $c_v^{\rm ion}$, $c_v^{\rm el}$, $c_{\rm s}$, $v_{\rm F}$ are given.

6. Physical discussion and summary

For convenience we shall, in this section, again revert back to the language appropriate for a thin long bar, for the change to a thin slab of any shape can be adapted from section 5.

It is well known [8] that, for $k_{\rm B}T \ll \epsilon_{\rm F}$, $c_v^{\rm el} = \frac{\pi^2}{2}n_{\rm e}k_{\rm B}(\frac{T}{T_{\rm F}})$ where $T_{\rm F} \sim 8\times10^4$ K. Also, for $T \ll \theta_{\rm D} = \frac{\hbar c_{\rm s}}{k_{\rm B}}(6\pi^2 n_{\rm ion})^{\frac{1}{3}}$, the Debye specific heat $c_v^{\rm ion} \simeq \frac{12\pi^4}{5}n_{\rm ion}k_{\rm B}(\frac{T}{\theta_{\rm D}})^3$. In the temperature range of $T \lesssim 10$ K, $T/T_{\rm F} \sim 10^{-3} \sim (T/\theta_{\rm D})^3$ we see that $c_v^{\rm el} \simeq c_v^{\rm ion}$. It follows that, since $\gamma_{\rm ion} \lesssim 2$ and $\gamma_{\rm el} = 2/3$, (i.e., $\gamma_{\rm ion} \sim \gamma_{\rm el}$) the metallic thermal expansion coefficient [5]

$$\alpha_{\rm metal} = (\gamma_{\rm ion}c_v^{\rm ion} + \gamma_{\rm el}c_v^{\rm el})/3B \tag{31}$$

will have comparable contributions to the amount of thermal expansion from the ionic (the phonon) part and the electron part, except that the former depends on T^3 while the latter depends linearly on T. On the other hand, it is significant to note that, due to the factor $1/c_{\rm s}$ in the ionic contribution and the factor $1/v_{\rm F}$ in the electronic contribution to $\eta_{\rm metal}$ of equation (26), the *speed* of thermal expansion of a thin long bar of length L given by

$$v_{\rm exp} = \frac{\Delta L}{t_L^{\rm metal}} = \frac{\eta_{\rm metal}\alpha_{\rm metal}L\Delta T}{2} \gtrsim \frac{\eta_{\rm ph}\alpha_{\rm metal}L\Delta T}{2} \tag{32}$$

$$\simeq \frac{3\gamma_{\rm ion}c_v^{\rm ion}\alpha_{\rm metal}(\Delta T)^2}{c_{\rm s}\rho} \tag{33}$$

is actually dominated by the phonon contribution in this temperature range. Hypothetically, if an insulating bar composed purely of ions and a metallic bar composed purely of electrons were heated up side by side, we might be surprised to find that the sluggish phonons in the pure ion bar should finish their race of bar expansion much sooner (by a factor of $\sim10^2$) than the speedy electrons in the other bar, in spite of the fact that the final amounts of expansion of the two bars might be roughly the same. This ratio would become even greater at higher temperatures. It all stems from the two equations (34) and (35) below

$$\Delta t_L^{\rm ion} = \frac{2}{\eta_{\rm ion}} = \frac{1}{9}\left(\frac{L}{c_{\rm s}}\right)\frac{1}{\alpha_{\rm ion}\Delta T} = \frac{1}{9}\left(\frac{L}{c_{\rm s}}\right)\left(\frac{L}{\Delta L_{\rm ion}}\right) \tag{34}$$

$$\Delta t_L^{\rm el} = \frac{2}{\eta_{\rm el}} = \frac{4}{15\sqrt{3}}\left(\frac{L}{c_{\rm s}}\right)\left(\frac{v_{\rm F}}{c_{\rm s}}\right)\frac{1}{\alpha_{\rm el}\Delta T}$$

$$= \frac{4}{15\sqrt{3}}\left(\frac{L}{c_{\rm s}}\right)\left(\frac{L}{\Delta L_{\rm el}}\right)\left(\frac{v_{\rm F}}{c_{\rm s}}\right) \tag{35}$$

which will be physically explained later. We also note that since $\eta_{\rm metal} \propto \Delta T/L$, the speed $v_{\rm exp} = \Delta L/t_L^{\rm metal}$ is seen,

from equation (32), to depend only on the intrinsic material parameters in addition to ΔT, but *not* on the geometric parameters such as the length L or the shape of the expanding specimen. It is thus tempting to conjecture that this result could be extended so that, as the different parts of a specimen of *any* given geometric shape (not just a bar) expand individually as dictated by the same α_{metal}, the times required for the respective amounts of expansion would simply follow proportionately the linear distances from the center of mass of that shape. This conjecture has actually been proved in section 5. In the limit of $\Delta T \to 0$, equation (32) shows $v_{\text{exp}} \to 0$ as intuitively expected.

When the forward physical momentum $\vec{p}$ of an acoustic phonon (or a free electron) is reflected elastically at the expanding free end of the bar due to a temperature rise ΔT, its magnitude together with its energy would suffer a change caused by the Doppler shift. This shift, that appears as a reduction of the phonon (electron) momentum, is usually perturbatively proportional to the momentum itself as long as the fractional change is small. The corresponding reduction in the individual phonon (electron) energy is what microscopically gives rise to the energy dissipation that is accompanied by the damping of the forward phonon (electron) motion. Similarly, when a whole bunch of such phonons (electrons) rushes forward towards, and is subsequently reflected by, the expanding free end, the Doppler correction appears now as a negative change of the phonon (electron) *pressure* that is perturbatively proportional to the prevailing phonon (electron) *pressure* itself. Again the accompanying energy dissipation leads to the damping of the expanding motion of the macroscopic long thin bar. This is the physical content of equation (26). It also follows that, in the hypothetical race, the faster electrons suffer a perturbatively smaller fractional loss ($\propto v_{\text{end}}/v_{\text{F}}$) of energy from the Doppler correction than the slower phonons ($\propto v_{\text{end}}/c_{\text{s}}$), hence resulting in $(\eta_L^{\text{el}}/\eta_L^{\text{ion}})^{-1} = \Delta t_L^{\text{el}}/\Delta t_L^{\text{ion}} \sim v_{\text{F}}/c_{\text{s}}$ of equations (34) and (35).

One indication from equation (26) might seem puzzling. As $\Delta T \to 0$, the expansion time $\Delta t_L^{\text{metal}} \propto 1/\Delta T \to \infty$! Actually this is as it should be. For $\Delta T \to 0$ means $\Delta P = (\Delta P_{\text{ph}} + \Delta P_{\text{el}}) \propto \Delta T \to 0$, leading eventually to the vanishing of the inhomogeneous term on the right side of the differential equation (25). This, in turn, leads to an ever oscillating behavior of $\langle x(t)\rangle$ with frequency ω_{e} and an amplitude determined by the initial condition. In a word, it would never be damped, implying $\Delta t_L^{\text{metal}} \to \infty$. Only when $\Delta T \neq 0$ would the Doppler correction have a chance to damp the oscillations into a finite $\langle x(t)\rangle$ as $t \to \infty$ [1].

Some possibly related works concerning 'extended irreversible thermodynamics' [9–11] and some other works concerning radiation pressure resulting from wave propagation [12], as well as radiation forces associated with heat propagation [13], have recently been brought to our attention. A brief review of some of these works has been included in [1]. We shall, however, postpone a more detailed investigation of the relation of these to our present work for the near future.

Acknowledgments

The author is especially grateful to Dr Y Shvyd'ko for numerous correspondence on many aspects, experimental and others, in relation to the time of thermal expansion.

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